

## Preface to Grades Eight Through Twelve

The standards for grades eight through twelve are organized differently from those for kindergarten through grade seven. (A complete description of this organization is provided on page 78, “Introduction to Grades Eight Through Twelve.”) In grades eight through twelve, the mathematics studied is organized according to disciplines such as algebra and geometry. *Local educational agencies may choose to teach high school mathematics in a traditional sequence of courses (Algebra I, geometry, Algebra II, and so forth) or in an integrated fashion in which some content from each discipline is taught each year.*

However mathematics courses are organized, the core content of these subjects must be covered by the end of the sequences of courses, and all academic standards for achievement must be the same. The core content and the areas of emphasis are delineated in the discussions of the individual disciplines presented in this section.

What follows in this preface is a discussion of key standards and discipline-level emphases for Algebra I, geometry, Algebra II, and probability and statistics. These same disciplines will be tested under the statewide Standardized Testing and Reporting (STAR) program, which will offer both traditional discipline-based versions and integrated versions of its test. The following section describes standards for the academic content by discipline, along with the areas of emphasis in each discipline; it is not an endorsement of a particular choice of structure for courses or a particular method of teaching the mathematical content. The additional advanced subjects of mathematics covered in the standards (linear algebra, advanced placement probability and statistics, and calculus) are not discussed in this section because many of these advanced subjects are not taught in every middle school or high school. Schools and districts may combine the subject matter of these various disciplines. Many combinations of these subjects are possible, and this framework does not prescribe a single instructional approach.

By the eighth grade, students’ mathematical sensitivity should be sharpened. Students should start perceiving logical subtleties and appreciating the need for sound mathematical arguments before making conclusions. As students progress in the study of mathematics, they learn to understand the meaning of logical implication; test general assertions; realize that one counterexample is enough to show that a general assertion is false; conceptually understand that the truth of a general assertion in a few cases does not allow the conclusion that it is true in all cases; distinguish between something being proven and a mere plausibility argument; and identify logical errors in chains of reasoning.

However mathematics courses are organized, all academic standards for achievement must be the same.

From kindergarten through grade seven, these standards have impressed on the students the importance of logical reasoning in mathematics. Starting with grade eight, students should be ready for the basic message that logical reasoning is the underpinning of all mathematics. In other words, every assertion can be justified by logical deductions from previously known facts. Students should begin to learn to *prove* every statement they make. Every textbook or mathematics lesson should try to convey this message and to convey it well.

## Mathematical Proofs

A misapprehension in mathematics education is that proofs occur only in Euclidean geometry and that elsewhere one merely learns to *solve problems* and *do computations*. Problem solving and symbolic computations are nothing more than different manifestations of mathematical proofs. To illustrate this point, the following discussion shows how the usual computations leading to the solution of a simple linear equation are nothing but the steps of a well-disguised proof of a theorem.

Consider the problem of solving this equation:

$$x - \frac{1}{4}(3x - 1) = 2x - 5$$

Multiply both sides by 4 to get:

$$4x - (3x - 1) = 8x - 20$$

Then simplify the left side to get:

$$x + 1 = 8x - 20$$

Transposing  $x$  from left to right yields:

$$1 = 7x - 20$$

One more transposition, and a division, gives the result  $x = 3$ .

This would seem to be an entirely mechanical procedure that involves no proof because both the hypothesis and conclusion are hidden.

Closer examination reveals that what is really being stated is a mathematical theorem:

A number  $x$  satisfies  $x - \frac{1}{4}(3x - 1) = 2x - 5$   
when and only when  $x = 3$ .

That  $x = 3$  satisfies the equation is easy to see. The less trivial part of the preceding theorem is the assertion that if a number  $x$  satisfies  $x - \frac{1}{4}(3x - 1) = 2x - 5$ , then  $x$  is necessarily equal to 3. A *proof* of this fact is presented next in a two-column format:

$$1. \quad x - \frac{1}{4}(3x - 1) = 2x - 5$$

1. Hypothesis

$$2. \quad 4(x - \frac{1}{4}(3x - 1)) = 4(2x - 5)$$

2.  $a = b$  implies  $ca = cb$  for all numbers  $a, b, c$ .

$$3. \quad 4x - 4(\frac{1}{4}(3x - 1)) = 4(2x) - 20$$

3. Distributive law

**Grades Eight  
Through  
Twelve**

Starting with grade eight, students should be ready for the basic message that logical reasoning is the underpinning of all mathematics.

- |   |  |
|---|--|
| 4. $4x - (4 \cdot \frac{1}{4})(3x - 1) = (4 \cdot 2)x - 20$ | 4. Associative law for multiplication  |
| 5. $4x - (3x - 1) = 8x - 20$                                | 5. $1 \cdot a = a$ for all numbers $a$ .   |
| 6. $4x + (-3x + 1) = 8x - 20$                               | 6. $-(a - b) = (-a + b)$ for all numbers $a, b$ .  |
| 7. $(4x + (-3x)) + 1 = 8x - 20$                             | 7. Associative law for addition  |
| 8. $x + 1 = 8x - 20$  | 8. $4x + (-3x) = (4 + (-3))x$ , by the distributive law.   |
| 9. $-x + (x + 1) = -x + (8x - 20)$                          | 9. Equals added to equals are equal.   |
| 10. $(-x + x) + 1 = (-x + 8x) - 20$                         | 10. Associative law for addition<br>$0 + 1 = 1$ .  |
| 11. $1 = 7x - 20$   | 11. $-x + 8x = (-1 + 8)x$ , by the distributive law.   |
| 12. $1 + 20 = (7x - 20) + 20$                               | 12. Equals added to equals are equal.  |
| 13. $21 = 7x + [(-20) + 20]$                                | 13. Associative law for addition   |
| 14. $21 = 7x$   | 14. $-a + a = 0$ for all $a$ ; $b + 0 = b$ for all $b$ .   |
| 15. $3 = x$   | 15. Multiply (14) by $\frac{1}{7}$ and apply the associative law and multiplicative inverse to $\frac{1}{7}(7x)$ . |
| 16. $x = 3$ .   | 16. $a = b$ implies that $b = a$ Q.E.D.  |

The purpose of giving this proof is by no means to suggest that, in school mathematics, linear equations should ever be solved in this tedious fashion. Rather, the intention is to show that even certain standard procedures that students tend to take for granted are nevertheless mathematical proofs in disguise. Furthermore, without the realization that such a mathematical proof is lurking behind the well-known formalism of solving linear equations, an author of an algebra textbook or a teacher in a classroom would most likely emphasize the wrong points in the presentation of beginning algebra.

The preceding proof clearly exposes the need for *generality* in the presentation of the associative laws and distributive law. In these standards these laws are taught starting with grade two, but it is probably difficult to convince students that such seemingly obvious statements deserve discussion. For example, if one has to believe that  $3(5 + 11) = 3 \cdot 5 + 3 \cdot 11$ , all one has to do is to expand both sides: clearly,  $3 \cdot 16 = 15 + 33$  because both sides are equal to 48. However, one look at the deduction of step 3 from step 2 in the preceding mathematical

Without the realization that a mathematical proof is lurking behind the well-known formalism of solving linear equations, a teacher would most likely emphasize the wrong points in the presentation of beginning algebra.

demonstration would make it clear that the hands-on approach to the distributive law is useless in this situation. Begin with the right-hand side of the equation:

$$4(2x - 5) = 4(2x) - 4 \cdot 5.$$

Here  $x$  is an *arbitrary* number, so we are not saying that

$$4(2 \cdot 17 - 5) = 4(2 \cdot 17) - 4 \cdot 5$$

or that

$$4(2 \cdot 172 - 5) = 4(2 \cdot 172) - 4 \cdot 5.$$

Were that the case, the equality could again have been verified by expanding both expressions. Rather, the assertion is that, *although we do not know what number  $x$  is*, nevertheless it is true that  $4(2x - 5) = 4(2x) - 4 \cdot 5$ . There is no alternative except to justify this general statement by using a general rule: the distributive law. The same comment applies to the other applications of the associative laws and the distributive law in the preceding proof.

It must be recognized that some proofs may not be accessible until the later grades, such as the reason for the formula of the circumference of a circle,  $C = 2\pi r$ . Nevertheless, *every* technique taught in mathematics is nothing but proofs in disguise. The validity of this statement can be revealed by considering a special case, such as this word problem for grade eight:

Jan had a bag of marbles. She gave one-half to James and then one-third of the marbles still in the bag to Pat. She then had 6 marbles left. How many marbles were in the bag to start with? (TIMSS, gr. 7–8, N-16)

The solution to the problem follows:

Suppose Jan had  $n$  marbles to start with. If she gave one-half to James, then she had  $\frac{n}{2}$  marbles left. According to the problem, she then gave one-third of what was left to Pat (i.e., she gave  $(\frac{1}{3}) \cdot (\frac{1}{2})n$  to Pat). Thus she gave  $(\frac{1}{6})n$  marbles to Pat, and what she had left was  $(\frac{1}{2})n - (\frac{1}{6})n = (\frac{1}{3})n$ . But the problem states that Jan had “6 marbles left.” So  $(\frac{1}{3})n = 6$ , and  $n = 18$ . Therefore, Jan had 18 marbles to begin with.

The next step is to analyze in what sense the preceding solution masks a proof. First, the usual solution as presented previously can be broken into two distinct steps:

1. *Setting up the equation:* If  $n$  is the number of marbles Jan had to begin with, then the given data imply:

$$(n - (\frac{1}{2})n) - (\frac{1}{3})(n - (\frac{1}{2})n) = 6.$$

2. *Solving the equation:* This step requires the proof of the following theorem:

$n$  satisfies the equation  $(\frac{1}{2})n - (\frac{1}{3})(\frac{1}{2})n = 6$  when and only when  $n = 18$ .

Step 1 and step 2 exemplify the two components of mathematics in grades eight through twelve: teaching the skills needed to transcribe sometimes untidy raw data into mathematical terms and teaching the skills needed to draw precise logical conclusions from clearly stated hypotheses. Neither can be slighted.

Grades Eight  
Through  
Twelve

Every technique taught  
in mathematics  
is nothing but  
proofs in disguise.

## Misconceptions in Mathematics Problems

It should be pointed out, however, that the built-in uncertainty and indeterminacy of step 1—which can lead to the setting up of several distinct equations and hence several distinct solutions—has led to the view of mathematics as an imprecise discipline in which a problem may have more than one correct answer. *This lack of understanding of the sharp distinction between step 1 and step 2 has had the deleterious effect of downgrading the importance of obtaining a single correct answer and jettisoning the inherent precision of mathematics.* As a result the rigor and precision needed for step 2 have been vigorously questioned. Such a misconception of mathematics would never have materialized had the process of transcription been better understood. This level of rigor and precision is embedded in the standards and is essential so that all students can develop mathematically to the level required in the *Mathematics Content Standards*.

The following is an extreme example of the kind of misconception discussed earlier:

The 20 percent of California families with the lowest annual earnings pay an average of 14.1 percent in state and local taxes, and the middle 20 percent pay only 8.8 percent. What does that difference mean? Do you think it is fair? What additional questions do you have?

Any attempt to solve this problem would require a missing definition in mathematical terms of how to decide what is “fair,” and consideration of much unspecified information about taxes and society. Since it is impossible to transcribe the problem as stated into mathematics, step 1 (setting up the equation) cannot be carried out, and so there can be no step 2 (solving the equation). This example is therefore not a mathematical problem. Hence, the fact that it has no single correct answer can in no way lend credence to the assertion that mathematics is uncertain or imprecise.

The preceding discussion explains that mathematical proofs are the underpinning of all of mathematics. Beginning with grade eight, students must deepen their understanding of the essential foundations for reasoning provided by mathematical proofs. It would be counterproductive to force every student to write a two-column proof at every turn, and it would be equally foolish to require all mathematics instructional materials to be as pedantic about giving such details as the two-column proof shown earlier in this preface. Nevertheless, the message that proofs underlie everything being taught should be clear in the instructional material and mathematical lessons taught in grades eight through twelve. In particular, all instructional materials—not just those for geometry, but especially those for algebra and trigonometry—should carefully present proofs of mathematical assertions when the situation calls for them. For example, an algebra textbook which asserts that a polynomial  $p(x)$  satisfying  $p(a) = 0$  for some number  $a$  must contain  $x - a$  as a factor, but which does not offer a detailed proof beyond a few concrete examples for corroboration, is not presenting material compatible with the standards.

# Algebra I

In algebra, students learn to reason symbolically, and the complexity and types of equations and problems that they are able to solve increase dramatically as a consequence. The key content for the first course, Algebra I, involves understanding, writing, solving, and graphing linear and quadratic equations, including systems of two linear equations in two unknowns. Quadratic equations may be solved by factoring, completing the square, or applying the quadratic formula. Students should also become comfortable with operations on monomial and polynomial expressions. They learn to solve problems employing all of these techniques, and they extend their mathematical reasoning in many important ways, including justifying steps in an algebraic procedure and checking algebraic arguments for validity.

## Transition from Arithmetic to Algebra

Perhaps the fundamental difficulty for many students making the transition from arithmetic to algebra is their failure to recognize that the symbol  $x$  stands for a number. For example, the equation  $3(2x - 5) + 4(x - 2) = 12$  simply means that a certain number  $x$  has the property that when the arithmetic operations  $3(2x - 5) + 4(x - 2)$  are performed on it as indicated, the result is 12. The problem is to find that number (solution). Teachers can emphasize this point by having students perform a series of arithmetic computations (using pen and paper) starting with  $x = 1$ ,  $x = 2$ ,  $x = 3$ ,  $x = 4$ , and so forth, thereby getting  $-13$ ,  $-3$ ,  $7$ ,  $17$ , and so forth. These computations show that none of  $1$ ,  $2$ ,  $3$ ,  $4$  can be that solution. Going from  $x = 3$  to  $x = 4$ , the value of the expression changes from  $7$  to  $17$ ; therefore, it is natural to guess that the solution would be between  $3$  and  $4$ . More experimentation eventually gives  $3.5$  as the solution.

Working backwards, since  $3(2(3.5) - 5) + 4((3.5) - 2) = 12$ , one can apply the distributive law and commutative and associative laws to unwind the expression, intentionally not multiplying out  $2(3.5)$ ,  $4(3.5)$ , and so forth, to get:

$$3.5 = \frac{12 + 3(5) + 2(4)}{3(2) + 4}.$$

But this is exactly the principle of solving the equation  $3(2x - 5) + 4(x - 2) = 12$  for the number  $x$ :

$$x = \frac{12 + 3(5) + 2(4)}{3(2) + 4}.$$

Perhaps the fundamental difficulty for many students making the transition from arithmetic to algebra is their failure to recognize that the symbol  $x$  stands for a number.

One can bring closure to such a lesson by stressing the similarity between the handling of the algebraic equation and the earlier simple arithmetic operations.

## Basic Skills for Algebra I

The first basic skills that must be learned in Algebra I are those that relate to understanding linear equations and solving systems of linear equations. In Algebra I the students are expected to solve only two linear equations in two unknowns, but this is a basic skill. The following six standards explain what is required:

- 4.0** Students simplify expressions before solving linear equations and inequalities in one variable, such as  $3(2x - 5) + 4(x - 2) = 12$ .
- 5.0** Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.
- 6.0** Students graph a linear equation and compute the  $x$ - and  $y$ -intercepts (e.g., graph  $2x + 6y = 4$ ). They are also able to sketch the region defined by linear inequalities (e.g., they sketch the region defined by  $2x + 6y < 4$ ).
- 7.0** Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.
- 9.0** Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.
- 15.0** Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

Each of these standards can be a source of difficulty for students, but they all reflect basic skills that must be understood so that students can advance to the next level in their understanding of mathematics. Moreover, modern applications of mathematics rely on solving systems of linear equations more than on any other single technique that students will learn in kindergarten through grade twelve mathematics. Consequently, it is essential that they learn these skills well.

## Point-Slope Formula

Perhaps the most perplexing difficulty that students have is with Standard 7.0. It often seems very hard for them to understand this point. But it is one of the most critical skills in this section. In particular, the following idea must be clearly understood before the students can progress further: A point lies on a line given by, for example, the equation  $y = 7x + 3$  if and only if the coordinates of that point  $(a, b)$  satisfy the equation when  $x$  is replaced with  $a$  and  $y$  with  $b$ . One way of explaining this idea is to emphasize that the graph of the equation  $y = 7x + 3$  is

precisely the set of points  $(a, b)$  for which replacing  $x$  by  $a$  and  $y$  by  $b$  gives a true statement. (For example,  $(3, 2)$  is not on the graph because replacing  $x$  with 3 and  $y$  with 2 gives the statement  $2 = 23$ , which is not true.) Thus, the graph consists of all points of the form  $(a, 7a + 3)$ . It also follows from these considerations that the root  $r$  of the linear polynomial  $7x + 3$  is the  $x$ -intercept of the graph of  $y = 7x + 3$  because  $(r, 0)$  is on the graph.

An additional comment about Standard 7.0 is that, although it singles out the point-slope formula, it is understood that students also have to know how to write the equation of a line when two of its points are given. However, the fact that the slope of a line is the same regardless of which pair of points on the line are used for its definition depends on the considerations of similar triangles. (This fact is first mentioned in Algebra and Functions Standard 3.3 for grade seven.) This small gap in the logical development should be made clear to students, with the added assurance that they will learn the concept in geometry. The same comment applies also to the fact that two nonvertical lines are perpendicular if and only if the product of their slopes is  $-1$  (Standard 8.0).

## Quadratic Equations

The next basic topic is the development of an understanding of the structure of quadratic equations. Here, one repeats the considerations involved in linear equations, such as graphing and understanding what it means for a point  $(x, y)$  to be on the graph. In particular, the graphical interpretation of finding the zeros of a quadratic equation by identifying the  $x$ -intercepts with the graph is very important and, as was the case with linear equations, is also a source of serious difficulty. Equally important is the recognition that if  $a, b$  are the roots of a quadratic polynomial, then up to a multiplicative constant, it is equal to  $(x - a)(x - b)$ .

When the discriminant of a quadratic polynomial is negative, the quadratic formula yields no information at this point because students have not yet been introduced to complex numbers. This deficiency will be remedied in Algebra II. The following standards show which skills students in a first-year algebra course need for solving quadratic equations. Among these, Standards 14.0 and 19.0 on the use of completing the square to prove the quadratic formula are basic.

- 14.0** Students solve a quadratic equation by factoring or completing the square.
- 19.0** Students know the quadratic formula and are familiar with its proof by completing the square.
- 20.0** Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.
- 21.0** Students graph quadratic functions and know that their roots are the  $x$ -intercepts.
- 23.0** Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.

The next basic topic is the development of an understanding of the structure of quadratic equations.

## Additional Comments

Students should be carefully guided through the solving of word problems by using symbolic notations. Many students may be so overwhelmed by the symbolic notation that they start to manipulate symbols carelessly, and word problems become incomprehensible. Teachers and publishers need to be sensitive to this difficulty. In addition to Standard 15.0, cited previously, the other relevant standards for solving word problems using symbolic notations are:

- 10.0** Students add, subtract, multiply and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.
- 13.0** Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.

Among the word problems of this level, those involving *direct* and *inverse* proportions occupy a prominent place. These concepts, which are often mired in the language of “proportional thinking,” need clarification. A quantity  $P$  is said to be *proportional to* another quantity  $Q$  if the quotient  $\frac{P}{Q}$  is a fixed constant  $k$ . This  $k$  is then called the *constant of proportionality*. Students should be made aware that this is a mathematical definition, and there is no need to look for linguistic subtleties concerning the phrase “to be proportional to.” Similarly,  $P$  is said to be *inversely proportional to*  $Q$  if the product  $PQ$  is equal to a fixed nonzero constant  $h$ .

In Standard 13.0 the emphasis should be on *formal* rational expressions in a number  $x$  instead of on rational *functions*. Many of these formal techniques will become increasingly important in Algebra II and trigonometry. The rules of exponents, for example, are fundamental to an understanding of the exponential and logarithmic functions. Many students fail to cope with the latter topics because their understanding of the rules of (fractional) exponents is weak. The skills in the following standards need to be emphasized in a first-year algebra course:

- 2.0** Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.
- 12.0** Students simplify fractions with polynomials in the numerator and denominator by factoring both and reducing them to the lowest terms.

The gist of Standards 16.0 through 18.0 is to introduce students to a precise concept of functions in the language of ordered pairs. Introducing this concept needs to be done gently because students at this stage of their mathematical development may not be ready for this level of abstraction. However, during a

first-year algebra course is the stage at which students should see and use the functional notation  $f(x)$  for the first time.

In Standard 24.0 students begin to learn simple logical arguments in algebra. They can be taught the proof that square roots of prime numbers are never rational, thereby solidifying to a certain extent their understanding of rational and irrational numbers (grade seven, Number Sense Standard 1.4). In Standard 3.0 students are taught to solve equations and inequalities involving absolute values, but it is not necessary to introduce the interval notation  $[a, b]$ ,  $(a, b)$ ,  $[a, b)$ , and so forth at this point. However, they should be introduced to the set notation  $\{a, b, c, \dots\}$  and  $\{x: x \text{ satisfies property } P\}$  and to the empty set  $\emptyset$  in, for example, Standard 17.0. Finally, students should become familiar with the terminology “solution set” of Standard 9.0—meaning the set of all solutions.

**Chapter 3**  
Grade-Level  
Considerations

Grades Eight  
Through  
Twelve

**Algebra I**

During a first-year algebra course is the stage at which students should see and use the functional notation  $f(x)$  for the first time.

## Geometry

The main purpose of the geometry curriculum is to develop geometric skills and concepts and the ability to construct formal logical arguments and proofs in a geometric setting. Although the curriculum is weighted heavily in favor of plane (synthetic) Euclidean geometry, there is room for placing special emphasis on coordinated geometry and its transformations.

The first standards introduce students to the basic nature of logical reasoning in mathematics:

- 1.0** Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning.
- 3.0** Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

Starting with undefined terms and axioms, students learn to establish the validity of other assertions through logical deductions; that is, they learn to prove theorems. This is their first encounter with an axiomatic system, and experience shows that they do not easily adjust to the demand of total precision needed for the task. In general, it is important to impress on students from the beginning that the main point of a proof is the mathematical correctness of the argument, not the literary polish of the writing or the adherence to a particular proof format.

### Inductive Reasoning

Standard 1.0 also calls for an understanding of inductive reasoning. Students are expected to recognize that inductive reasoning by itself does not prove anything in mathematics, but that it fosters the kind of intuition that is indispensable for finding proofs. To this end students should be encouraged to draw many pictures to develop a geometric sense and to amass a wealth of geometric data in the process. Many students—including high-achieving ones—complete a course in geometry with so little geometric intuition that, given three noncollinear points, they cannot even begin to visualize what the circumcircle of these points must be like. One way to develop this geometric sense is to have the students become familiar with the basic straightedge-compass constructions, as illustrated in the following standard:

- 16.0** Students perform basic constructions with a straightedge and compass, such as angle bisectors, perpendicular bisectors, and the line parallel to a given line through a point off the line.

Students should be encouraged to draw many pictures to develop a geometric sense and to amass a wealth of geometric data in the process.

It would be desirable to introduce students to these constructions early in the course and leave the proofs of their validity to the appropriate place of the logical development later.

## Geometric Proofs

The subject then turns to geometric proofs in earnest. The foundational results of plane geometry are embodied in the following standards:

- 2.0** Students write geometric proofs, including proofs by contradiction.
- 4.0** Students prove basic theorems involving congruence and similarity.
- 7.0** Students prove and use theorems involving the properties of parallel lines cut by a transversal, the properties of quadrilaterals, and the properties of circles.
- 12.0** Students find and use measures of sides and of interior and exterior angles of triangles and polygons to classify figures and solve problems.
- 21.0** Students prove and solve problems regarding relationships among chords, secants, tangents, inscribed angles, and inscribed and circumscribed polygons of circles.

Grades Eight  
Through  
Twelve

**Geometry**

It has become customary in high school geometry textbooks to start with axioms that incorporate real numbers. Although doing geometric proofs with real numbers runs counter to the spirit of Euclid, this approach is a good mathematical compromise in the context of school mathematics. However, the parallel postulate occupies a special place in geometry and should be clearly stated in the traditional form: Through a point not on a given line  $L$ , there is exactly one line parallel to  $L$ . Because this postulate played a fundamental role in the development of mathematics up to the nineteenth century, the significance of the postulate should be discussed. And because there always exists at least one parallel line through a point to a given line, the import of this postulate lies in the uniqueness of the parallel line. A discussion of this postulate provides a natural context to show students the key concept of uniqueness in mathematics—a concept that experience indicates students usually find difficult.

The parallel postulate occupies a special place in geometry and should be clearly stated in the traditional form.

One should soft-pedal the early theorems that are the immediate deductions from the axioms, regardless of which axiomatic system is used. These deceptively simple theorems are in fact conceptually difficult and pedagogically deadly. It is better to proceed to the proofs of more advanced, and therefore more substantive, theorems. (See Appendix C, “Resource for Secondary School Teachers: Circumcenter, Orthocenter, and Centroid.”) It is also recommended that the topics of circles and similarity be taught as early as possible. Once those topics have been presented, the course enters a new phase not only because of the interesting theorems that can now be proved but also because the concept of similarity

expands the applications of algebra to geometry. These applications might include determining one side of a regular decagon on the unit circle through the use of the quadratic formula as well as the applications of geometry to practical problems.

It is often not realized that theorems for circles can be introduced very early in a geometry course. For instance, the remarkable theorem that inscribed angles on a circle which intercept equal arcs must be equal can in fact be presented within three weeks after the introduction of axioms. All it takes is to prove the following two theorems:

1. Base angles of isosceles triangles are equal.
2. The exterior angle of a triangle equals the sum of opposite interior angles.

At this point it is necessary to deal with one of the controversies in mathematics education concerning the format of proofs. It has been argued that the traditional two-column format is stultifying for students and that the format for proofs in the mathematics literature is always paragraph proofs. While the latter observation is true, teachers should be aware that a large part of the reason for using paragraph proofs is the expense of typesetting more elaborate formats, not that paragraph proofs are intrinsically better or clearer. In fact, neither of these claims of superiority for paragraph proofs is actually valid. Furthermore, it appears that for beginners to learn the precision of argument needed, the two-column format is best. After the students have shown a mastery of the basic logical skills, it would be appropriate to relax the requirements on form. *But the teacher should never relax the requirement that all arguments presented by the students be precise and correct.*

## Pythagorean Theorem

One of the high points of elementary mathematics, in fact of all of mathematics, is the Pythagorean theorem:

### 14.0 Students prove the Pythagorean theorem.

This theorem can be proved initially by using similar triangles formed by the altitude on the hypotenuse of a right triangle. Once the concept of area is introduced (Standard 8.0), students can prove the Pythagorean theorem in at least two more ways by using the familiar picture of four congruent right triangles with legs  $a$  and  $b$  nestled inside a square of side  $a + b$ .

### 8.0 Students know, derive, and solve problems involving the perimeter, circumference, area, volume, lateral area, and surface area of common geometric figures.

### 10.0 Students compute areas of polygons, including rectangles, scalene triangles, equilateral triangles, rhombi, parallelograms, and trapezoids.

For rectilinear figures in the plane, the concept of area is simple because everything reduces to a union of triangles. However, the course must deal with circles, and here limits must be used and the number  $\pi$  defined. The concept of limit can be employed intuitively without proofs. If the area or length of a circle is defined

as the limit of approximating, inscribing, or circumscribing regular polygons, then  $\pi$  is either the area of a disk of unit radius or the ratio of circumference to diameter, and heuristic arguments (see the glossary) for the equivalence of these two definitions would be given.

The concept of volume, in contrast with that of area, is not simple even for polyhedra and should be touched on only lightly and intuitively. However, the formulas for volumes and surface areas of prisms, pyramids, cylinders, cones, and spheres (Standard 9.0) should be memorized.

An important aspect of teaching three-dimensional geometry is to cultivate students' spatial intuition. Most students find spatial visualization difficult, which is all the more reason to make the teaching of this topic a high priority.

The basic mensuration formulas for area and volume are among the main applications of geometry. However, the Pythagorean theorem and the concept of similarity give rise to even more applications through the introduction of trigonometric functions. The basic trigonometric functions in the following standards should be presented in a geometry course:

**18.0** Students know the definitions of the basic trigonometric functions defined by the angles of a right triangle. They also know and are able to use elementary relationships between them. For example,  $\tan(x) = \sin(x)/\cos(x)$ ,  $(\sin(x))^2 + (\cos(x))^2 = 1$ .

**19.0** Students use trigonometric functions to solve for an unknown length of a side of a right triangle, given an angle and a length of a side.

Finally, the Pythagorean theorem leads naturally to the introduction of rectangular coordinates and coordinate geometry in general. A significant portion of the curriculum can be devoted to the teaching of topics embodied in the next two standards:

**17.0** Students prove theorems by using coordinate geometry, including the midpoint of a line segment, the distance formula, and various forms of equations of lines and circles.

**22.0** Students know the effect of rigid motions on figures in the coordinate plane and space, including rotations, translations, and reflections.

## The Connection Between Algebra and Geometry

These standards lead students to the next level of sophistication: an algebraic and transformation-oriented approach to geometry. Students begin to see how algebraic concepts add a new dimension to the understanding of geometry and, conversely, how geometry gives substance to algebra. Thus straight lines are no longer merely simple geometric objects; they are also the graphs of linear equations. Conversely, solving simultaneous linear equations now becomes finding the point of intersection of straight lines. Another example is the interpretation

Students begin to see how algebraic concepts add a new dimension to the understanding of geometry and, conversely, how geometry gives substance to algebra.

of the geometric concept of congruence in the Euclidean plane as a correspondence under an isometry of the coordinate plane. Concrete examples of isometries are studied: rotations, reflections, and translations. It is strongly suggested that the discussion be rounded off with at least the precise statement of the structure theorem: Every isometry of the coordinate plane is a translation or the composition of a translation and a rotation or the composition of a translation, a rotation, and a reflection.

*Special attention should be given to the fact that a gap in Algebra I must be filled here.* Standards 7.0 and 8.0 of Algebra I assert that:

1. The graph of a linear equation is a straight line.
2. Two straight lines are perpendicular if and only if their slopes have a product of  $-1$ .

These facts should now be proved.

### Additional Comments and Cautionary Notes

This section provides further comments and cautions in presenting the material in geometry courses.

**Introduction to proofs.** An important point to make to students concerning proofs is that while the written proofs presented in class should serve as models for exposition, *they should in no way be a model of how proofs are discovered.* The perfection of the finished product can easily mislead students into thinking that they must likewise arrive at their proofs with the same apparent ease. Teachers need to make clear to their students that the actual thought process is usually full of false starts and that there are many zigzags between promising leads and dead ends. Only trial and error can lead to a correct proof.

This awareness of the nature of solving mathematical problems might lead to a deemphasis of the rigid requirements on the writing of two-column proofs in some classrooms.

**Students' perceptions of proofs.** The first part of the course sets the tone for students' perceptions of proofs. With this in mind, it is advisable to discuss, mostly *without proofs*, those first consequences of the axioms that are needed for later work. A few proofs should be given for illustrative purposes; for example, the equality of vertical angles or the equality of the base angles of an isosceles triangle and its converse. There are two reasons for the recommendation to begin with only a few proofs. The foremost is that a complete logical development is neither possible nor desirable. This has to do with the intrinsic complexity of the structure of Euclidean geometry (see Greenberg 1993, 1–146). A second reason is the usual misconception that such elementary proofs are easy for beginners. Working on the level of axioms is actually more difficult for beginners than working with the theorems that come a little later in the logical development. This difficulty occurs because, on the one hand, working with axioms requires a heavy reliance on formal logic without recourse to intuition—in fact often *in spite of* one's intuition. On the other hand, working on the level of axioms does not usually have a clear direction or goal, and it is difficult to convince students to learn

Working on the level of axioms is actually more difficult for beginners than working with the theorems that come a little later in the logical development.

something without a clearly stated goal. If one so desires, students can always be made to go back to prove the elementary theorems *after* they have already developed a firm grasp of proof techniques.

**Structured work with proofs.** Students' first attempts at proofs need to be structured with care. At the beginning of the development of this skill, instead of asking students to do *many* trivial proofs after showing them the proofs of two or three easy theorems, it might be a good strategy to proceed as follows:

1. As early as possible, the students might be shown a generous number of proofs of substantive theorems so that they can gain an understanding of what a proof is before they write any proofs themselves.
2. As a prelude to constructing proofs themselves, the students might provide reasons for some of the steps in the sample (substantive) proofs instead of constructing extremely easy proofs on their own.
3. After an extended exposure to nontrivial proofs, students might be asked to give proofs of simple corollaries of substantive theorems.

The reason for steps 2 and 3 is to make students, from the beginning, associate proofs with real mathematics rather than perform a formal ritual. This goal can be accomplished with the use of *local axiomatics*; that is, if the proof of a theorem makes use of facts not previously proved, let these facts be stated clearly before the proof. These facts need not be previously proven but should ideally be sufficiently plausible even without a proof. Extensive use of local axiomatics would make possible, sufficiently early in the course, the presentation of interesting but perhaps advanced theorems. *In Appendix C the ideas in steps 2 and 3 are put to use to demonstrate how they might work.*

**Development of geometric intuition.** The following geometric constructions are recommended to develop students' geometric intuition. (In this context *construction* means "construction with straightedge and compass.") It is understood that all of them will be proved at some time during the course of study. The constructions that students should be able to do are:

- Bisecting an angle
- Constructing the perpendicular bisector of a line segment
- Constructing the perpendicular to a line from a point on the line and from a point not on the line
- Duplicating a given angle
- Constructing the parallel to a line through a point not on the line
- Constructing the circumcircle of a triangle
- Dividing a line segment into  $n$  equal parts
- Constructing the tangent to a circle from a point on the circle
- Constructing the tangents to a circle from a point not on the circle
- Locating the center of a given circle
- Constructing a regular  $n$ -gon on a given circle for  $n = 3, 4, 5, 6$

**Use of technology.** This is the place to add a word about the use of technology. The availability of good computer software makes the accurate drawing of geometric figures far easier. Such software can enhance the experience of making

Grades Eight  
Through  
Twelve

### Geometry

Students' first  
attempts at  
proofs need to  
be structured  
with care.

the drawings in the constructions described previously. In addition, the ease of making accurate drawings encourages the formulation and exploration of geometric conjectures. For example, it is now easy to convince oneself that the intersections of adjacent angle trisectors of the angles of a triangle are most likely the vertices of an equilateral triangle (Morley's theorem). If students do have access to such software, the potential for a more intense mathematical encounter is certainly there. In encouraging students to use the technology, however, one should not lose sight of the fact that the excellent visual evidence thus provided must never be taken as a replacement for understanding. For example, software may give the following heuristic evidence for why the sum of the angles of a triangle is  $180^\circ$ . When any three points on the screen are clicked, a triangle with these three points as vertices appears. When each angle is clicked again, three numbers will appear that give the angle measurement of each angle. When these numbers are added,  $180^\circ$  will be the answer. Furthermore, no matter the shape of the triangle, the result will always be the same.

While such exercises may boost one's belief in the validity of the theorem about the sum of the angles, it must be recognized that these angle measurements have added nothing to one's understanding of why this theorem is true. Furthermore, if one really wants to have a hands-on experience with angle measurements in order to check the validity of this theorem, the best way is to do it painstakingly by hand on paper. Morley's theorem, mentioned earlier, is another illustration of the same principle: evidence cannot replace proofs. The computer program would not reveal *the reason* the three points are always the vertices of an equilateral triangle.

**Introduction to the coordinate plane.** Students should know that the coordinate plane provides a concrete example that satisfies all the axioms of Euclidean geometry if the lines are defined as the graphs of linear equations  $ax + by = c$ , with at least one of  $a$  and  $b$  not equal to zero. Lines  $a_1x + b_1y = c_1$  and  $a_2x + b_2y = c_2$  are defined as parallel if  $(a_1, b_1)$  is proportional to  $(a_2, b_2)$ , but  $(a_1, b_1, c_1)$  is not proportional to  $(a_2, b_2, c_2)$ . The verification of the axioms is straightforward.

# Algebra II

**A**lgebra II expands on the mathematical content of Algebra I and geometry. There is no single unifying theme. Instead, many new concepts and techniques are introduced that will be basic to more advanced courses in mathematics and the sciences and useful in the workplace. In general terms the emphasis is on abstract thinking skills, the function concept, and the algebraic solution of problems in various content areas.

## Absolute Value and Inequalities

The study of absolute value and inequalities is extended to include simultaneous linear systems; it paves the way for linear programming—the maximization or minimization of linear functions over regions defined by linear inequalities. The relevant standards are:

- 1.0** Students solve equations and inequalities involving absolute value.
- 2.0** Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or with matrices.

The concept of Gaussian elimination should be introduced for  $2 \times 2$  matrices and simple  $3 \times 3$  ones. The emphasis is on concreteness rather than on generality. Concrete applications of simultaneous linear equations and linear programming to problems in daily life should be brought out, but there is no need to emphasize linear programming at this stage. While it would be inadvisable to advocate the use of graphing calculators all the time, such calculators are helpful for graphing regions in connection with linear programming once the students are past the initial stage of learning.

## Complex Numbers

At this point of students' mathematical development, knowledge of complex numbers is indispensable:

- 5.0** Students demonstrate knowledge of how real and complex numbers are related both arithmetically and graphically. In particular, they can plot complex numbers as points in the plane.
- 6.0** Students add, subtract, multiply, and divide complex numbers.

From the beginning it is important to stress the geometric aspect of complex numbers; for example, the addition of two complex numbers can be shown in terms of a parallelogram. And the key difference between real and complex numbers should be pointed out: The complex numbers cannot be linearly ordered in the same way as real numbers are (the real *line*).

## Polynomials and Rational Expressions

The next general technique is the *formal* algebra of polynomials and rational expressions:

- 3.0** Students are adept at operations on polynomials, including long division.
- 4.0** Students factor polynomials representing the difference of squares, perfect square trinomials, and the sum and difference of two cubes.
- 7.0** Students add, subtract, multiply, divide, reduce, and evaluate rational expressions with monomial and polynomial denominators and simplify complicated rational expressions, including those with negative exponents in the denominator.

The importance of formal algebra is sometimes misunderstood. The argument against it is that it has insufficient real-world relevance and it leads easily to an overemphasis on mechanical drills. There seems also to be an argument for placing the study of exponential function ahead of polynomials in school mathematics because exponential functions appear in many real-world situations (compound interest, for example). There is a need to affirm the primacy of polynomials in high school mathematics and the importance of formal algebra. The potential for abuse in Standard 3.0 is all too obvious, but such abuse would be realized only if the important ideas implicit in it are not brought out. These ideas all center on the abstraction and hence on the generality of the formal algebraic operations on polynomials. Thus the division algorithm (long division) leads to the understanding of the roots and factorization of polynomials. The factor theorem, which states that  $(x-a)$  divides a polynomial  $p(x)$  if and only if  $p(a) = 0$ , should be proved; and students should know the proof. The rational root theorem could be proved too, but only if there is enough time to explain it carefully; otherwise, many students would be misled into thinking that *all* the roots of a polynomial with integer coefficients are determined by the divisibility properties of the first and last coefficients.

It would be natural to first prove the division algorithm and the factor theorem for polynomials with real coefficients. But it would be vitally important to revisit both and to point out that the same proofs work, verbatim, for polynomials with *complex* coefficients. This procedure not only provides a good exercise on complex numbers but also nicely illustrates the built-in generality of formal algebra.

Two remarks about Standard 7.0 are relevant: (1) a rational expression should be treated formally, and its function-theoretic aspects (the domain of definition,

for example) need not be emphasized at this juncture; and (2) fractional exponents of polynomials and rational expressions should be carefully discussed here.

## Quadratic Functions

The first high point of the course is the study of quadratic (polynomial) functions:

- 8.0** Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.
- 9.0** Students demonstrate and explain the effect that changing a coefficient has on the graph of quadratic functions; that is, students can determine how the graph of a parabola changes as  $a$ ,  $b$ , and  $c$  vary in the equation  $y = a(x-b)^2 + c$ .
- 10.0** Students graph quadratic functions and determine the maxima, minima, and zeros of the function.

What distinguishes Standard 8.0 from the same topic in Algebra I is the newly acquired generality of the quadratic formula: It now solves all equations  $ax^2 + bx + c = 0$  with real  $a$ ,  $b$ , and  $c$  regardless of whether or not  $b^2 - 4ac < 0$ , and it does so even when  $a$ ,  $b$ , and  $c$  are *complex* numbers. Again it should be stressed that the purely *formal* derivation of the quadratic formula makes it valid for any object  $a$ ,  $b$ , and  $c$  as long as the usual arithmetic operations on numbers can be applied to them. In particular, it makes no difference whether the numbers are real or complex. This premise illustrates the built-in generality of formal algebra. Students need to know every aspect of the proof of the quadratic formula. They should also be made aware (1) that with the availability of complex numbers, any quadratic polynomial  $ax^2 + bx + c = 0$  with real or complex  $a$ ,  $b$ , and  $c$  can be factored into a product of two linear polynomials with complex coefficients; (2) that  $c$  is the product of the roots and  $-b$  is their sum; and (3) that if  $a$ ,  $b$ , and  $c$  are real and the roots are complex, then the roots are a conjugate pair.

Standard 9.0 brings the study of quadratic polynomials to a new level by regarding them as a function. This new point of view leads to the exact location of the maximum, minimum, and zeros of this function by use of the quadratic formula (or, more precisely, by completing the square) without recourse to calculus. The practical applications of these results are as important as the theory.

Another application of completing the square is given in Standard 17.0, through which students learn, among other things, how to bring a quadratic polynomial in  $x$  and  $y$  without an  $xy$  term to standard form and recognize whether it represents an ellipse or a hyperbola.

Grades Eight  
Through  
Twelve

Algebra II

Students need to know every aspect of the proof of the quadratic formula.

## Logarithms

A second high point of Algebra II is the introduction of two of the basic functions in all of mathematics:  $e^x$  and  $\log x$ .

- 11.0** Students prove simple laws of logarithms.
  - 11.1 Students understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.
  - 11.2 Students judge the validity of an argument according to whether the properties of real numbers, exponents, and logarithms have been applied correctly at each step.
- 12.0** Students know the laws of fractional exponents, understand exponential functions, and use these functions in problems involving exponential growth and decay.
- 15.0** Students determine whether a specific algebraic statement involving rational expressions, radical expressions, or logarithmic or exponential functions is sometimes true, always true, or never true.

The theory should be done carefully, and students are responsible for the proofs of the laws of exponents for  $a^m$  where  $m$  is a rational number and of the basic properties of  $\log_a x$ :  $\log_a (x_1 x_2) = \log_a x_1 + \log_a x_2$ ,  $\log_a (\frac{1}{x}) = -\log_a x$ , and  $\log_a x^r = r \log_a x$ , where  $r$  is a rational number (Standard 15.0). The functional relationships  $\log_a (a^x) = x$  and  $a^{\log(t)} = t$ , where  $a$  is the base of the log function in the second equation, should be taught without a detailed discussion of inverse functions in general, as students are probably not ready for it yet. Practical applications of this topic to growth and decay problems are legion.

## Arithmetic and Geometric Series

A third high point of Algebra II is the study of arithmetic and geometric series:

- 23.0** Students derive the summation formulas for arithmetic series and for both finite and infinite geometric series.

The geometric series, finite and infinite, is of great importance in mathematics and the sciences, physical and social. Students should be able to recognize this series under all its guises and compute its sum with ease. In particular, they should know by heart the basic identity that underlies the theory of geometric series:

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

This identity gives another example of the utility of formal algebra, and the identity is used in many other places as well (the differentiation of monomials, for example). It should be mentioned that while it is tempting to discuss the arithmetic and geometric series using the sigma notation

$$\sum_{i=1}^n,$$

it would be advisable to resist this temptation so that the students are not overburdened.

## Binomial Theorem

Students should learn the binomial theorem and how to use it:

- 20.0** Students know the binomial theorem and use it to expand binomial expressions that are raised to positive integer powers.
- 18.0** Students use fundamental counting principles to compute combinations and permutations.
- 19.0** Students use combinations and permutations to compute probabilities.

In this context the applications almost come automatically with the theory.

Finally, Standards 16.0 (geometry of conic sections), 24.0 (composition of functions and inverse functions), and 25.0 may be taken up if time permits.

Chapter 3  
Grade-Level  
Considerations

Grades Eight  
Through  
Twelve

**Algebra II**

The definition of the trigonometric functions is made possible by the notion of similarity between triangles.

## Trigonometry

**T**rigonometry uses the techniques that students have previously learned from the study of algebra and geometry. The trigonometric functions studied are defined geometrically rather than in terms of algebraic equations, but one of the goals of this course is to acquaint students with a more algebraic viewpoint toward these functions.

Students should have a clear understanding that the definition of the trigonometric functions is made possible by the notion of similarity between triangles.

A basic difficulty confronting students is one of superabundance: There are six trigonometric functions and seemingly an infinite number of identities relating to them. The situation is actually very simple, however. Sine and cosine are by far the most important of the six functions. Students must be thoroughly familiar with their basic properties, including their graphs and the fact that they give the coordinates of every point on the unit circle (Standard 2.0). Moreover, three identities stand out above all others:  $\sin^2 x + \cos^2 x = 1$  and the addition formulas of sine and cosine:

**3.0** Students know the identity  $\cos^2(x) + \sin^2(x) = 1$ :

- 3.1. Students prove that this identity is equivalent to the Pythagorean theorem (i.e., students can prove this identity by using the Pythagorean theorem and, conversely, they can prove the Pythagorean theorem as a consequence of this identity).
- 3.2. Students prove other trigonometric identities and simplify others by using the identity  $\cos^2(x) + \sin^2(x) = 1$ . For example, students use this identity to prove that  $\sec^2(x) = \tan^2(x) + 1$ .

**10.0** Students demonstrate an understanding of the addition formulas for sines and cosines and their proofs and can use those formulas to prove and/or simplify other trigonometric identities.

Students should know the proofs of these addition formulas. An acceptable approach is to use the fact that the distance between two points on the unit circle depends only on the angle between them. Thus, suppose that angles  $a$  and  $b$  satisfy  $0 < a < b$ , and let  $A$  and  $B$  be points on the unit circle making angles  $a$  and  $b$  with the positive  $x$ -axis. Then  $A = (\cos a, \sin a)$ ,  $B = (\cos b, \sin b)$ ,

and the distance  $d(A, B)$  from  $A$  to  $B$  satisfies the equation:

$$d(A, B)^2 = (\cos b - \cos a)^2 + (\sin b - \sin a)^2.$$

On the other hand, the angle from  $A$  to  $B$  is  $(b - a)$ , so that the distance from the point  $C = (\cos(b - a), \sin(b - a))$  to  $(1, 0)$  is also  $d(A, B)$  because the angle from  $C$  to  $(1, 0)$  is  $(b - a)$  as well. Thus:

$$d(A, B)^2 = (\cos(b - a) - 1)^2 + \sin^2(b - a).$$

Equating the two gives the formula:

$$\cos(b - a) = \cos a \cos b + \sin a \sin b.$$

From this formula both the sine and cosine addition formulas follow easily.

Students should also know the special cases of these addition formulas in the form of half-angle and double-angle formulas of sine and cosine (Standard 11.0). These are important in advanced courses, such as calculus. Moreover, the addition formulas make possible the rewriting of trigonometric sums of the form  $A \sin(x) + B \cos(x)$  as  $C \sin(x + D)$  for suitably chosen constants  $C$  and  $D$ , thereby showing that such a sum is basically a displaced sine function. This fact should be made known to students because it is important in the study of wave motions in physics and engineering.

Students should have a moderate amount of practice in deriving trigonometric identities, but identity proving is no longer a central topic.

Of the remaining four trigonometric functions, students should make a special effort to get to know tangent, its domain of definition  $(-\frac{\pi}{2}, \frac{\pi}{2})$ , and its graph (Standard 5.0). The tangent function naturally arises because of the standard:

- 7.0** Students know that the tangent of the angle that a line makes with the  $x$ -axis is equal to the slope of the line.

Because trigonometric functions arose historically from computational needs in astronomy, their practical applications should be stressed (Standard 19.0). Among the most important are:

- 13.0** Students know the law of sines and the law of cosines and apply those laws to solve problems.
- 14.0** Students determine the area of a triangle, given one angle and the two adjacent sides.

These formulas have innumerable practical consequences.

Complex numbers can be expressed in polar forms with the help of trigonometric functions (Standard 17.0). The geometric interpretations of the multiplication and division of complex numbers in terms of the angle and modulus should be emphasized, especially for complex numbers on the unit circle. Mention should be made of the connection between the  $n$ th roots of 1 and the vertices of a regular  $n$ -gon inscribed in the unit circle:

- 18.0** Students know DeMoivre's theorem and can give  $n$ th roots of a complex number given in polar form.

## Mathematical Analysis

This discipline combines many of the trigonometric, geometric, and algebraic techniques needed to prepare students for the study of calculus and other advanced courses. It also brings a measure of closure to some topics first brought up in earlier courses, such as Algebra II. The functional viewpoint is emphasized in this course.

### Mathematical Induction

The eight standards are fairly self-explanatory. However, some comments on four of them may be of value. The first is mathematical induction:

- 3.0** Students can give proofs of various formulas by using the technique of mathematical induction.

This basic technique was barely hinted at in Algebra II; but at this level, to understand why the technique works, students should be able to use the technique fluently and to learn enough about the natural numbers. They should also see examples of why the step to get the induction started and the induction step itself are both necessary. Among the applications of the technique, students should be able to prove by induction the binomial theorem and the formulas for the sum of squares and cubes of the first  $n$  integers.

### Roots of Polynomials

Roots of polynomials were not studied in depth in Algebra II, and the key theorem about them was not mentioned:

- 4.0** Students know the statement of, and can apply, the fundamental theorem of algebra.

This theorem should not be proved here because the most natural proof requires mathematical techniques well beyond this level. However, there are “elementary” proofs that can be made accessible to some of the students. In a sense this theorem justifies the introduction of complex numbers. An application that should be mentioned and proved on the basis of the fundamental theorem of algebra is that for polynomials with real coefficients, complex roots come in conjugate pairs. Consequently, all polynomials with real coefficients can be written as the product of real quadratic polynomials. The quadratic formula should be reviewed from the standpoint of this theorem.

## Conic Sections

The third area is conic sections (see Standard 5.0). Students learn not only the geometry of conic sections in detail (e.g., major and minor axes, asymptotes, and foci) but also the *equivalence* of the algebraic and geometric definitions (the latter refers to the definitions of the ellipse and hyperbola in terms of distances to the foci and the definition of the parabola in terms of distances to the focus and directrix). A knowledge of conic sections is important not only in mathematics but also in classical physics.

## Limits

Finally, students are introduced to limits:

- 8.0** Students are familiar with the notion of the limit of a sequence and the limit of a function as the independent variable approaches a number or infinity. They determine whether certain sequences converge or diverge.

This standard is an introduction to calculus. The discussion should be intuitive and buttressed by much numerical data. The calculator is useful in helping students explore convergence and divergence and guess the limit of sequences. If desired, the precise definition of *limit* can be carefully explained; and students may even be made to memorize it, but it should not be emphasized. For example, students can be taught to prove why for linear functions  $f(x)$ ,  $\lim_{x \rightarrow a} f(x) = f(a)$  for any  $a$ , but it is more likely a ritual of manipulating  $\epsilon$ 's and  $\delta$ 's in a special situation than a real understanding of the concept. The time can probably be better spent on other proofs (e.g., mathematical induction).

Chapter 3  
Grade-Level  
Considerations

Grades Eight  
Through  
Twelve

Mathematical  
Analysis

Mastery of this academic content will provide students with a solid foundation in probability and facility in processing statistical information.

## Probability and Statistics

This discipline is an introduction to the study of probability, interpretation of data, and fundamental statistical problem solving. Mastery of this academic content will provide students with a solid foundation in probability and facility in processing statistical information.

Some of the topics addressed review material found in the standards for the earlier grades and reflect that this content should not disappear from the curriculum. These topics include the material with respect to the common concepts of mean, median, and mode and to the various display methods in common use, as stated in these standards:

- 6.0 Students know the definitions of the *mean*, *median*, and *mode* of a distribution of data and can compute each in particular situations.
- 8.0 Students organize and describe distributions of data by using a number of different methods, including frequency tables, histograms, standard line and bar graphs, stem-and-leaf displays, scatterplots, and box-and-whisker plots.

In the early grades students also receive an introduction to probability at a basic level. The next topic will expand on this base so that students can find probabilities for multiple discrete events in various combinations and sequences. The standards in Algebra II related to permutations and combinations and the fundamental counting principles are also reflective of the content in these standards:

- 1.0 Students know the definition of the notion of *independent events* and can use the rules for addition, multiplication, and complementation to solve for probabilities of particular events in finite sample spaces.
- 2.0 Students know the definition of *conditional probability* and use it to solve for probabilities in finite sample spaces.
- 3.0 Students demonstrate an understanding of the notion of *discrete random variables* by using them to solve for the probabilities of outcomes, such as the probability of the occurrence of five heads in 14 coin tosses.

The most substantial new material in this discipline is found in Standard 4.0:

- 4.0** Students are familiar with the standard distributions (normal, binomial, and exponential) and can use them to solve for events in problems in which the distribution belongs to those families.

Instruction typically flows from the counting principles for discrete binomial variables to the rules for elaborating probabilities in binomial distributions. The fact that these probabilities are simply the terms in a binomial expansion provides a strong link to Algebra II and the binomial theorem. From this base, basic probability topics can be expanded into the treatment of these standard distributions. In the binomial case students should now be able to define the probability for a range of possible outcomes for a set of events based on a single-event probability and thus to develop better understanding of probability and density functions.

The normal distribution, which is the limiting form of a binomial distribution, is typically introduced next. Students are not to be expected to integrate this distribution, but they can answer probability questions based on it by referring to tabled values. Students need to know that the mean and the standard deviation are parameters for this distribution. Therefore, it is important to understand variance, based on averaged squared deviation, as an index of variability and its importance in normal distributions, as stated in these standards:

- 5.0** Students determine the mean and the standard deviation of a normally distributed random variable.
- 7.0** Students compute the variance and the standard deviation of a distribution of data.

Standard 4.0 also includes exponential distributions with applications, for example, in lifetime of service and radioactive decay problems. Including this distribution acquaints students with probability calculations for other types of processes. Here, students learn that the distribution is defined by a scale parameter, and they learn simple probability computations based on this parameter.